Chaos, noise, and synchronization reconsidered

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In a recent paper, Maritan and Banavar [Phys. Rev. Lett. **72**, 1451 (1994)] reported the synchronization of identical chaotic systems by additive noise. We relate such a synchronization to the maximum Lyapunov exponent of a single system and discuss the underlying mechanisms of the effect. In the case of the Lorenz equations, the nonvanishing mean of the noise mimics a parameter change leading to synchronization. For the logistic map a state dependence of the fluctuations is induced by the boundary conditions.

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I. INTRODUCTION

The effect of fluctuations on chaotic dynamics attracted much attention [1–10] since chaotic systems are, by definition, extremely sensitive to perturbations. Somewhat counterintuitive effects such as *noise-induced order* [11,12], *stabilization by noise* [13,14], or *stochastic resonance* [15,16] are of particular interest. These effects require specific properties of the chaotic system and the noise source: Noise-induced order was found in highly nonuniform systems [11,12], stabilization by noise is based on multiplicative fluctuations [13,14] , and stochastic resonance requires comparable time scales of periodic modulations and mean transition times [15,16]. Therefore, the findings of Maritan and Banavar [1] regarding the synchronization of chaos in the logistic map and in the Lorenz model due to additive noise appeared as a surprise and evoked some discussion [17,18], especially since earlier simulations of the logistic map with additive noise revealed robustness of the positive Lyapunov exponent [3,12].

The aim of this report is to resolve this contradiction. We will show that their observations are not genuine noise effects but a specific result of the bias of their noise terms. In the case of the Lorenz model, the nonvanishing mean of the noise term is solely responsible for the synchronization. For the logistic map, their noise terms are actually state dependent, which leads to negative *finite time Lyapunov exponents* explaining the synchronization.

II. LORENZ MODEL

First, we briefly recall the results of Ref. [1] concerning the Lorenz equations [19]

$$\begin{aligned}\frac{dx}{dt} &= 10(y-x),\\ \frac{dy}{dt} &= -xz+28x-y+R(t),\\ \frac{dz}{dt} &= xy-\frac{8}{3}z.\end{aligned} \tag{1}$$

The authors simulate two identical copies of these chaotic systems "linked with a common noise term" $R(t)$ [1]. As discussed in Refs. [6,20,21], the maximum Lyapunov exponent is defined via linearization along noisy trajectories, and thus it explicitly describes "the separation of nearby orbits subject to the same external noise [6]."

Let us denote trajectories of the two identical systems by $\vec{x}_1$ and $\vec{x}_2$:

$$\frac{d\vec{x}_1}{dt} = \vec{f}(\vec{x}_1) + \vec{R}(t), \tag{2}$$

$$\frac{d\vec{x}_2}{dt} = \vec{f}(\vec{x}_2) + \vec{R}(t). \tag{3}$$

If we introduce the difference $\vec{q} = \vec{x}_2 - \vec{x}_1$ we obtain easily

$$\frac{d\vec{q}}{dt} = J(\vec{x})\vec{q} + O(\vec{q}^{\,2}) \qquad \text{with} \quad J_{ij} = \frac{df_i}{dx_j}. \tag{4}$$

In this way, the dynamics of the difference is governed by the Jacobian of a single system. Consequently, a negative Lyapunov exponent implies synchronization of the two systems after some transient time as follows: There is a finite probability that the trajectories of the two systems come sufficiently close that linear stability analysis is applicable. Then, the difference decays exponentially due to the negative Lyapunov exponent. Consequently, the Lyapunov exponent of a single noisy system is the appropriate quantity to characterize the synchronization of identical systems [17], and therefore, we focus our attention on the effect of noise on Lyapunov exponents.

The authors of Ref. [1] simulate the random term $R(t)$ by adding every $\delta t = 0.001$ time units random numbers equidistributed in $[0, W\sqrt{\delta t}]$. Obviously, their noise term has a nonvanishing mean which explains immediately the synchronization for overcritical W: The noise mimics an additive constant in the equations. This additional term in the y equation leads to a stable focus, and hence, to negative Lyapunov exponents.

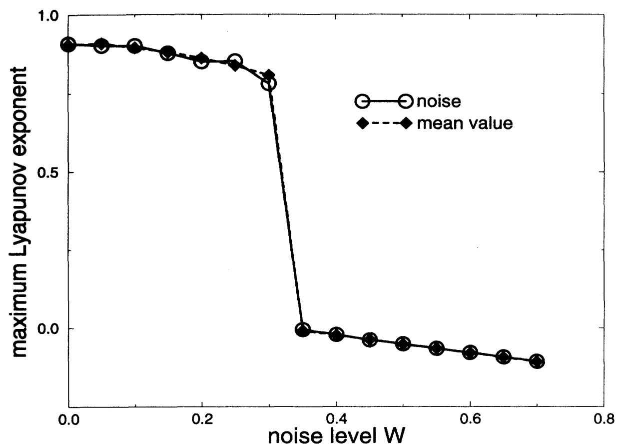

FIG. 1. Decrease of the maximum Lyapunov exponent for the Lorenz system. Circles, noise according to Ref. [1]; diamonds, noise replaced by its mean value.

Figure 1 demonstrates that virtually the same decrease of the Lyapunov exponent is obtained if the fluctuations are replaced by the mean value of the noise term.

It was already briefly mentioned in Ref. [1] that *symmetric noise* gives no synchronization. This confirms our point of view that not the random component but merely the nonvanishing mean stabilizes.

III. LOGISTIC MAP

The interpretation of synchronization of logistic maps is less straightforward since the Lyapunov exponent remains positive as also pointed out by Pikovsky [17]. Let us recall the simulations presented in [1]. The authors consider a pair of fully developed logistic maps

$$x_{n+1} = 4x_n(1 - x_n) + R_n. \tag{5}$$

The random number R_n is chosen from the interval $[-W, W]$ with the constraint $0 < x_{n+1} < 1$, i.e., if R_n violates the bounds a new random number is chosen. The authors find that the orbits of two such maps coalesce for overcritical noise strength W if they are linked with the same realization of noise. Coalescence means that the distance of the orbits shrinks below a threshold ε. As argued by Pikovsky [17], this is not a *stabilization* in a sense of a negative Lyapunov exponent but an effect of finite precision ε. Nevertheless, as demonstrated in the reply [18], there is a remarkably sharp transition to synchronization around $W = 0.5$. The explanation of this transition is the aim of the remainder of this section.

For the following considerations, we decompose the map (5) into two steps. First, we consider the deterministic mapping $y_n := 4x_n(1 - x_n)$. Then the random term R_n is added. Since only random numbers that keep the orbit in the unit interval are accepted, the noise acting on y_n is equidistributed in a reduced interval:

$$R_n \in [\max(-W, -y_n), \min(1 - y_n, W)]. \tag{6}$$

Consequently, the mean value of the noise is nonzero if y_n is sufficiently close to the boundaries 0 or 1. More precisely,

$$\langle R_n \rangle = 0 \quad \text{for} \quad y_n > W \quad \cap \quad 1 - y_n > W, \tag{7}$$

$$\langle R_n \rangle = \frac{W - y_n}{2} > 0 \quad \text{for} \quad y_n < W \quad \cap \quad 1 - y_n > W, \tag{8}$$

$$\langle R_n \rangle = \frac{1 - y_n - W}{2} < 0 \quad \text{for} \quad y_n > W \quad \cap \quad 1 - y_n < W, \tag{9}$$

$$\langle R_n \rangle = \tfrac{1}{2} - y_n \quad \text{for} \quad y_n < W \quad \cap \quad 1 - y_n < W. \tag{10}$$

In analogy to the procedure leading to Fig. 1, we substitute the random term R_n in the logistic equation by its mean value $\langle R_n(y_n) \rangle$ [cf. Eqs. (7)–(10)]. Figure 2 shows that chaos disappears for increasing W as in the case of the Lorenz model. This indicates that the bias of the noise plays the central role.

However, we emphasize that the replacement of the noise by the deterministic term $\langle R_n(x_n) \rangle$ defines quite a different dynamical system. In order to analyze the actual stochastic process, we discuss in the following the effect of the fluctuations on the invariant density and Lyapunov exponents.

In the deterministic case, the evolution of the density is governed by the Frobenius-Perron equation

$$\rho_{n+1}(x) = [\mathcal{U}\rho_n](x) = \frac{\rho_n(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}) + \rho_n(\frac{1}{2} + \frac{1}{2}\sqrt{1-x})}{4\sqrt{1-x}}, \tag{11}$$

with the stationary solution ρ^0

$$\rho^0(x) = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{12}$$

Thus, for $W = 0$ the invariant density has singularities

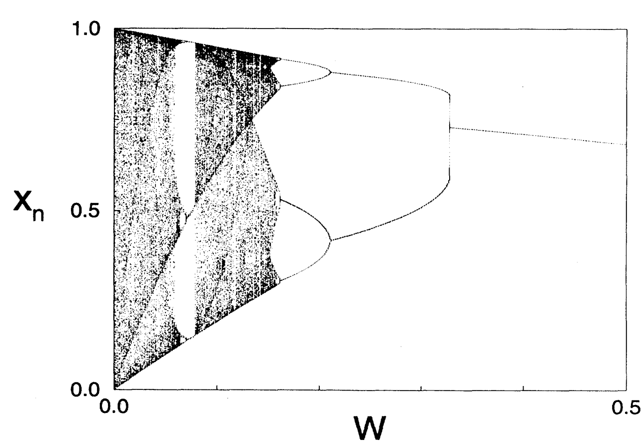

FIG. 2. Bifurcation diagram of the logistic equation (5) in which the random term is replaced by its bias $\langle R_n(y_n) \rangle$.

at the boundaries $x = 0$ and $x = 1$, which are the most unstable regions ($|\frac{dx_{n+1}}{dx_n}|$ attains its maximum). The corresponding Lyapunov exponent is given by

$$\lambda = \int_0^1 \frac{\ln|4-8x|}{\pi\sqrt{x(1-x)}}\, dx = \ln 2. \tag{13}$$

The noise term R_n induces transitions from the deterministic iterate $y := 4x(1-x)$ to neighboring values x. The corresponding evolution equation for the density reads

$$\rho_{n+1}(x) = \int_0^1 w(x|y)\ [\mathcal{U}\rho_n](y)\ dy. \tag{14}$$

The transition probabilities $w(x|y)$ can be explicitly written as

$$w(x|y) = \frac{\chi_{[0,1]}(x)\ \chi_{[x-w,x+w]}(y)}{c(x,W)} \tag{15}$$

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{for} \quad x \in [a,b], \\ 0 & \text{for} \quad x \notin [a,b], \end{cases} \tag{16}$$

where the functions $\chi_{[]}$ reflect the boundary conditions. Note that the normalization constant $c(x,W)$ depends on noise amplitude and the state x. The state dependence of the random term leads to a repulsion of orbits from the boundary, and consequently, the probability density is less concentrated at 0 and 1. Particularly for $W > 1$, an equidistribution results:

$$\rho^0(x) \equiv 1 \tag{17}$$

with a Lyapunov exponent $\lambda = \ln 4 - 1$. As already mentioned, the changes of the density do not give negative Lyapunov exponents for any W, which could immediately explain coalescence of orbits. At this point, we are reminded that coalescence in Ref. [1] means that the distance shrinks below ε. This may happen even by chance, and the remaining question is: Why are such events found for $W \approx 0.6$ even after as few as 10^5 iterations? The answer can be extracted from finite-time Lyapunov exponents as discussed in [6] and [12]:

$$\lambda^{(m)} = \frac{1}{m}\sum_{i=1}^{m} \ln|4-8x_i|.$$

Figure 3 shows histograms of these averaged (over $m = 10$ iterations) expansion rates.

For $W = 0$, the density is concentrated around the mean value $\ln 2$, whereas for $W = 0.6$ a pronounced tail towards negative values can be seen. Note, that $\lambda^{(10)} = -1.5$ implies a contraction by a factor of $e^{-15} \approx 3\times10^{-7}$.

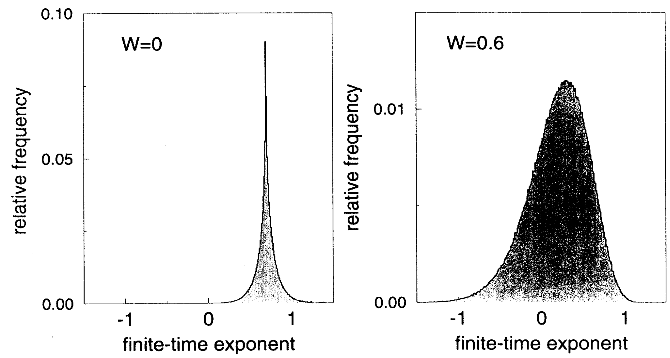

FIG. 3. Normalized histograms of expansion rates for $W = 0$ (left) and $W = 0.6$ (right) from 10^7 iterations.

This drastic change of the stability properties is induced by the bias of the random term as discussed above. Since random numbers are discarded if they violate the bounds $0 < x_{n+1} < 1$, there is a net bias towards $x = 0.5$, where the slope $\frac{dx_{n+1}}{dx_n} = 4 - 8x$ is zero.

IV. CONCLUSIONS

Effects of additive noise on Lyapunov exponents were studied in various systems [3,5–9]. It turned out that, close to bifurcations [2,7,10] and for chaotic windows [4,6], noise tends to amplify chaoticity. At parameter values with chaotic dynamics, Lyapunov exponents are robust against small fluctuations in most cases [3,6,12].

It has been shown in this paper that also the synchronization found in Ref. [1] constitutes no exception, i.e., additive noise *per se* does not affect Lyapunov exponents significantly. The effect of synchronization was traced back to quite simple mechanisms. For the Lorenz model, we have shown that not the random part of $R(t)$ but merely its nonvanishing mean value induces synchronization. For the logistic map, the seemingly additive noise exhibits a significant dependence on x_n, i.e., it can be regarded as *state dependent noise* which is known to strongly affect the stationary probability density [22]. Indeed, we could show that the noise together with the boundary conditions moves orbits towards the middle of the unit interval where stability dominates. Therefore, the density of finite-time Lyapunov exponents exhibits a significant tail towards negative values, which makes synchronization in the sense of Ref. [1] possible.

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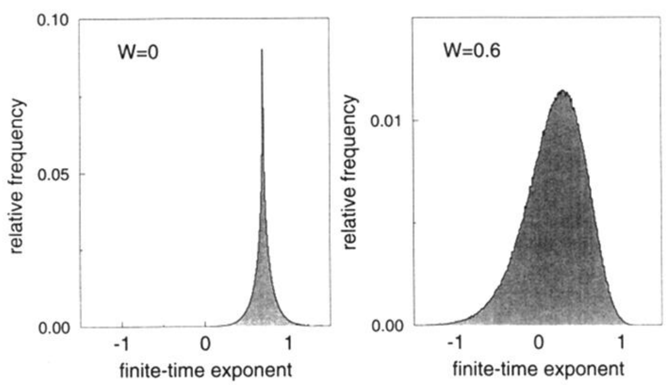

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